J. Math. Anal. Appl. 345 (2008) 805–815

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

A non-local boundary value problem method for parabolic equations backward in time

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ARTICLE INFO

Article history:
Received 2 January 2008
Available online 1 May 2008
Submitted by V. Radulescu

Keywords:
Parabolic equations backward in time
Ill-posed problems
Regularization
Non-local boundary value problems

ABSTRACT

The ill-posed parabolic equation backward in time

$$\begin{cases} u_t + Au = 0, & 0 < t < T, \\ \|u(T) - f\| \leqslant \epsilon \end{cases}$$

subject to the constraint

$$\|u(0)\| \leqslant E$$

with the positive self-adjoint unbounded operator A and $E > \epsilon > 0$ being given is regularized by the well-posed non-local boundary value problem

$$\begin{cases} u_t + Au = 0, & 0 < t < T, \\ \alpha u(0) + u(T) = f, & \alpha > 0. \end{cases}$$

The error estimates of Hölder type of the regularized solutions are obtained. These estimates improve the related results by Mel'nikova, Denche and Bessila.

1. Introduction

Let H be a Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, $A : D(A) \subset H \to H$ be a self-adjoint operator on H such that $-A$ generates a compact contraction semi-group $\{S(t)\}_{t\geqslant 0}$ on H. Let $\epsilon < E$ be two given positive numbers. For a positive real number T, consider the problem of finding a function $u : [0, T] \to H$ such that

$$\begin{cases} u_t + Au = 0, & 0 < t < T, \\ \|u(T) - f\| \leqslant \epsilon \end{cases} \tag{1.1}$$

subject to the constraint

$$\|u(0)\| \leqslant E \tag{1.2}$$

for f in H. This problem is well known to be severely ill-posed and regularization methods for it are required.

There have been several regularization methods for (1.1)–(1.2) such as the quasi-reversibility method [8], the method of Sobolev equations [5–7,11], the method of perturbation of the equation [10], Tikhonov regularization [15], the method

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.04.064

of non-local boundary value problems [1,3,4,9,12], etc. For a recent survey on backward parabolic equations, we refer the reader to [2].

In this note we follow Showalter [12], Clark and Oppenheimer [3] and Mel'nikova [9] in regularizing the problem (1.1)–(1.2) by the non-local boundary value problem

$$\begin{cases} u_t + Au = 0, & 0 < t < T, \\ \alpha u(0) + u(T) = f, & \alpha > 0. \end{cases} \tag{1.3}$$

We improve some of their error estimates and simplify some of their proofs. We note that Denche and Bessila [4] approximated the problem (1.1)–(1.2) by the problem

$$\begin{cases} u_t + Au = 0, & 0 < t < T, \\ -\alpha u_t(0) + u(T) = f, & \alpha > 0. \end{cases} \tag{1.4}$$

They obtained an error estimate at $t = 0$ of logarithmic type with a strong condition that $\|Au(0)\|$ is bounded. It means $u(0)$ has to be in the domain of A that is not frequently met in practice. We will show that we do not need to require u_t exist at $t = 0$ as these authors required but again by the problem (1.3) we can establish stability estimates which are comparable to theirs. Namely, assume that A admits an orthonormal eigenbasis $\{\phi_i\}_{i \geqslant 1}$ in H, associated with the eigenvalues $\{\lambda_i\}_{i \geqslant 1}$ such that

$$0 < \lambda_1 < \lambda_2 < \cdots, \quad \text{and} \quad \lim_{i \to +\infty} \lambda_i = +\infty.$$

If instead of (1.2) we have the stronger condition

$$\sum_{n=1}^{\infty} \lambda_n^{2\beta} \big(u(0), \phi_n\big)^2 \leqslant E_1^2 \tag{1.5}$$

with some positive constants β and E_1, then we get an error estimate of Hölder type in $(0, T)$ and logarithmic type at $t = 0$.

If instead of (1.2) we have the stronger condition

$$\sum_{n=1}^{\infty} e^{2\beta\lambda_n} \big(u(0), \phi_n\big)^2 \leqslant E_2^2 \tag{1.6}$$

with some positive constants β and E_2, then we have an error estimate of Hölder type in $[0, T)$.

2. The main results

From now on, for clarity, we denote the solution of (1.1)–(1.2) by $u(t)$, and the solution of the problem (1.3) by $v(t)$.

Definition 1. A function $v : [0, T] \to H$ is called a solution of (1.3) if $v \in C^1((0, T), H) \cap C([0, T], H)$, $v(t) \in D(A)$, $\forall t \in (0, T)$, and satisfies $v_t + Av = 0$ in $(0, T)$ and the boundary value condition $\alpha v(0) + v(T) = f$.

The well-posedness of (1.3) has been proved, e.g. in [3].

Theorem 1. *The following inequality holds*

$$\big\|v(t) - u(t)\big\| \leqslant Q(t, \alpha)\big(\alpha^{t/T-1}\epsilon + \alpha^{t/T} E\big), \quad \forall t \in [0, T]. \tag{2.1}$$

If we choose $\alpha = \frac{\epsilon}{E}$, *then*

$$\big\|v(t) - u(t)\big\| \leqslant 2Q\left(t, \frac{\epsilon}{E}\right)\epsilon^{t/T} E^{1-t/T}, \quad \forall t \in [0, T]. \tag{2.2}$$

Here,

$$\begin{aligned}
&Q(t, \alpha) = \min\big\{H(t, \alpha), K(t)\big\}, \quad t \in [0, T], \\
&H(t, \alpha) := \sqrt{(t/T)^{t/T}(1 - t/T)^{1-t/T}}\,\sqrt{2\alpha + 1}^{\,-t/T} \in (0, 1), \quad \forall t \in (0, T),\ \forall \alpha > 0, \\
&H(0) = 1, \qquad H(T) = 1/\sqrt{2\alpha + 1}, \\
&K(t) := (t/T)^{t/T}(1 - t/T)^{1-t/T} \in (0, 1), \quad \forall t \in (0, T), \\
&K(0) = K(T) = 1.
\end{aligned}$$

Remark 1. We have $2Q(t,\alpha) \in [1,2]$ for all $t \in [0,T]$ and

$$2Q\left(\frac{T}{2},\alpha\right) \leqslant 2K\left(\frac{T}{2}\right) = 1,$$
$$2Q(T,\alpha) \leqslant 2H(T,\alpha) = \frac{2}{\sqrt{2\alpha+1}} < 2.$$

Remark 2. A referee pointed out to us that the best possible worst case error for identifying $u(t)$ under assumptions (1.1) and (1.2) is given by

$$\omega(\epsilon) = \epsilon^{t/T} E^{1-t/T}$$

and there are regularization methods that guarantee this error bound (see [14]). However, to establish this result the authors of [14] have to choose the regularization parameter α depending on t which tends to zero as t tends to T and to infinity as t tends to zero. It is impractical for numerical calculations. In our case if we choose the regularization parameter in the same manner, say, $\alpha = \frac{\epsilon}{E}\frac{1-t/T}{t/T}$, then, as it will be proved at the end of Section 3.1,

$$\left\| v(t) - u(t) \right\| \leqslant \epsilon^{t/T} E^{1-t/T}, \quad \forall t \in (0,T). \tag{2.3}$$

Such a choice, as noted above, is impractical.

Theorem 1 does not give any information about the continuous dependence of the solution of (1.1)–(1.2) at $t = 0$ on the data, as the condition (1.2) is too weak. To establish this, we suppose that either we have (1.5) or (1.6). We will see that with these assumptions stability estimates of logarithmic type and Hölder type at $t = 0$ are respectively guaranteed.

We assume that A admits an orthonormal eigenbasis $\{\phi_i\}_{i\geqslant 1}$ in H, associated with the eigenvalues $\{\lambda_i\}_{i\geqslant 1}$ such that

$$0 < \lambda_1 < \lambda_2 < \cdots, \quad \text{and} \quad \lim_{i\to+\infty} \lambda_i = +\infty.$$

Theorem 2. *Suppose that instead of* (1.2)*, we have* (1.5)*. Then for all* $t \in [0,T)$

$$\left\| u(t) - v(t) \right\| \leqslant \begin{cases} Q(t,\alpha)\alpha^{\frac{t}{T}-1}\epsilon + \alpha^{\frac{t}{T}}\left(\frac{T}{\ln((\frac{T\lambda_1 e}{\beta(t)})^{\beta(t)}/\alpha)}\right)^{\beta} C(t)^{\frac{t}{T}-1} E_1, & \text{if } 0 < \alpha < (\frac{T\lambda_1}{\beta(t)})^{\beta(t)}, \\ Q(t,\alpha)\alpha^{\frac{t}{T}-1}\epsilon + \alpha\left(\frac{e^{\lambda_1 T}}{\lambda_1^{\beta(t)} C(t)}\right)^{1-\frac{t}{T}} E_1, & \text{if } \alpha \geqslant (\frac{T\lambda_1}{\beta(t)})^{\beta(t)}, \end{cases}$$

where $\beta(t) = \frac{\beta T}{T-t}, \forall t \in [0,T)$, $C(t) = 1$ *if* $0 < \beta(t) < 1$ *and* $C(t) = 2^{1-\beta(t)}$ *if* $\beta(t) \geqslant 1$*. If we choose* $\alpha = \alpha_0 := \frac{\epsilon^{1-\delta}}{E_1}$ *with* $0 < \delta < 1$*, then for all* $t \in [0,T)$

$$\left\| u(t) - v(t) \right\| \leqslant \begin{cases} \epsilon^{\frac{t}{T}} E_1^{1-\frac{t}{T}} \left\{ Q(t,\alpha_0)\epsilon^{\delta(1-\frac{t}{T})} + \epsilon^{-\delta\frac{t}{T}}\left(\frac{T}{\ln((\frac{T\lambda_1 e}{\beta(t)})^{\beta(t)} E_1/\epsilon^{1-\delta})}\right)^{\beta} C(t)^{\frac{t}{T}-1} \right\}, & \text{if } 0 < \alpha_0 < (\frac{T\lambda_1}{\beta(t)})^{\beta(t)}, \\ \epsilon^{\frac{t}{T}} E_1^{1-\frac{t}{T}} \left\{ Q(t,\alpha_0)\epsilon^{\delta(1-\frac{t}{T})} + \epsilon^{1-\delta-\frac{t}{T}}\left(\frac{e^{\lambda_1 T}}{\lambda_1^{\beta(t)} C(t)}\right)^{1-\frac{t}{T}} E_1^{\frac{t}{T}-1} \right\}, & \text{if } \alpha_0 \geqslant (\frac{T\lambda_1}{\beta(t)})^{\beta(t)}. \end{cases} \tag{2.4}$$

Remark 3. Since

$$\lim_{\epsilon\to 0^+} \frac{\ln(E_1/\epsilon)}{\ln((\frac{T\lambda_1 e}{\beta(t)})^{\beta(t)} E_1/\epsilon^{1-\delta})} = \frac{1}{1-\delta},$$

it directly follows from (2.4) that with $\alpha = \alpha_0$, for ϵ small enough, there exists a positive constant C_1 such that

$$\left\| u(t) - v(t) \right\| \leqslant C_1 \epsilon^{(1-\delta)t/T} E_1^{1-t/T} \left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}, \quad \forall t \in [0,T).$$

Further, from the last estimates of the theorem, at $t = 0$ we have the estimates

$$\left\| u(0) - v(0) \right\| \leqslant \begin{cases} E_1\left\{\epsilon^{\delta} + \left(\frac{T}{\ln((\frac{T\lambda_1 e}{\beta})^{\beta} E_1/\epsilon^{1-\delta})}\right)^{\beta} C(0)^{-1}\right\}, & \text{if } 0 < \alpha_0 < (\frac{T\lambda_1}{\beta})^{\beta}, \\ E_1\left\{\epsilon^{\delta} + \epsilon^{1-\delta}\left(\frac{e^{\lambda_1 T}}{\lambda_1^{\beta} C(0)}\right) E_1^{-1}\right\}, & \text{if } \alpha_0 \geqslant (\frac{T\lambda_1}{\beta})^{\beta}. \end{cases}$$

Set $C = (\frac{T}{1-\delta})^{\beta} C(0)^{-1} E_1$. Suppose that $(\frac{T\lambda_1 e}{\beta})^{\beta} E_1 \geqslant 1$. Then

$$\begin{aligned} \ln\left(\left(\frac{T\lambda_1 e}{\beta}\right)^{\beta} E_1/\epsilon^{1-\delta}\right) &= \ln\left(\left(\frac{T\lambda_1 e}{\beta}\right)^{\beta} E_1\right) + \ln\frac{1}{\epsilon^{1-\delta}} \\ &\geqslant \ln\frac{1}{\epsilon^{1-\delta}} \\ &= (1-\delta)\ln\frac{1}{\epsilon}. \end{aligned}$$

Hence

$$\|u(0)-v(0)\| \leqslant \begin{cases} E_1\epsilon^{\delta}+C(\ln\frac{1}{\epsilon})^{-\beta}, & \text{if } 0<\alpha_0<(\frac{T\lambda_1}{\beta})^{\beta}, \\ E_1\epsilon^{\delta}+\epsilon^{1-\delta}(\frac{e^{\lambda_1 T}}{\lambda_1^{\beta}C(0)}), & \text{if } \alpha_0 \geqslant (\frac{T\lambda_1}{\beta})^{\beta}. \end{cases}$$

Thus, for $\beta=1$, our error estimate at $t=0$ is comparable to that of Denche and Bessila [4].

Remark 4. If β is given, we choose $\alpha=\frac{\epsilon}{E_1}[\frac{1}{T}\ln\frac{E_1}{\epsilon}]^{\beta}$, then by direct examination we see that with ϵ being small there exists a positive constant C_2 such that

$$\|v(t)-u(t)\| \leqslant C_2\epsilon^{t/T}E_1^{1-t/T}\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta(T-t)/T}, \quad \forall t\in[0,T).$$

Remark 5. The referee pointed out that the best possible worst case error for identifying $u(t)$ under assumptions (1.1) and (1.5) is of the order

$$\epsilon^{t/T}E_1^{1-t/T}\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta(T-t)/T}\big(1+o(1)\big)$$

and there are regularization methods that guarantee this order (see [13]). However, as in [14] to establish such estimates the author of [13] has to choose the regularization parameter α depending on t which tends to zero as t tends to T and to infinity as t tends to zero. Furthermore, α is chosen depending on β, which is in general not known in practice. Thus, such a choice is impractical for numerical calculations. In our case if we choose the regularization parameter in the same manner, say, $\alpha=\frac{\epsilon}{E_1}(\frac{1}{T}\ln\frac{E_1}{\epsilon})^{\beta+\delta}$ for $\delta>0$, then, as $\epsilon\to 0$,

$$\|u(0)-v(0)\| \leqslant E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\big(1+o(1)\big), \tag{2.5}$$

and if we choose $\alpha=\frac{1-t/T}{t/T}\frac{\epsilon}{E_1}[\frac{1}{T}\ln\frac{E_1}{\epsilon}]^{\beta}$, then, as $\epsilon\to 0$,

$$\|v(t)-u(t)\| \leqslant \epsilon^{t/T}E_1^{1-t/T}\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta(T-t)/T}\big(1+o(1)\big), \quad \forall t\in(0,T). \tag{2.6}$$

Such a choice, as noted above, is impractical.

The proof of this remark will be given at the end of Section 3.2.

Theorem 3. *Suppose that instead of* (1.2) *we have* (1.6)*. Then for all* $t\in[0,T)$

$$\|u(t)-v(t)\| \leqslant \begin{cases} Q(t,\alpha)\alpha^{t/T-1}\epsilon+\alpha^{(t+\beta)/T}E_2, & \text{if } 0<\beta<T-t, \\ Q(t,\alpha)\alpha^{t/T-1}\epsilon+\alpha E_2, & \text{if } \beta\geqslant T-t. \end{cases} \tag{2.7}$$

If we choose $\alpha=\alpha_1:=\frac{\epsilon^{1-\delta}}{E_2}$ *for* $0<\delta<1$*, then for* $t\in[0,T)$

$$\|u(t)-v(t)\| \leqslant \begin{cases} \epsilon^{t/T}E_2^{1-t/T}(Q(t,\alpha_1)\epsilon^{\delta(1-t/T)}+\epsilon^{(\beta(1-\delta)-\delta t)/T}E_2^{t/T-1}), & \text{if } 0<\beta<T-t, \\ \epsilon^{t/T}E_2^{1-t/T}(Q(t,\alpha_1)\epsilon^{\delta(1-t/T)}+\epsilon^{1-t/T-\delta}E_2^{t/T-1}), & \text{if } \beta\geqslant T-t. \end{cases} \tag{2.8}$$

3. Proofs of the main results

3.1. Proof of Theorem 1

First, we prove some auxiliary results.

Lemma 1. *If* $v(t)$ *is a solution of* (1.3)*, then*

$$\|f\|^2 \geqslant \alpha^2\|v(0)\|^2+(2\alpha+1)\|v(T)\|^2.$$

Proof. We have

$$\|f\|^2=\big(\alpha v(0)+v(T),\alpha v(0)+v(T)\big)=\alpha^2\|v(0)\|^2+\|v(T)\|^2+2\alpha\big(v(0),v(T)\big). \tag{3.1}$$

Set $h(t) := (v(t), v(T-t))$, $t \in [0, T]$. By direct calculation we see that $h'(t) = 0$, $\forall t \in (0, T)$. Therefore, h is a constant. This implies that $h(0) = h(T/2)$. Thus, $(v(0), v(T)) = \|v(T/2)\|^2$.

Set $g(t) := \|v(t)\|^2$, $t \in [0, T]$. Then $g'(t) = -2(Av(t), v(t)) \leqslant 0$, $\forall t \in (0, T)$. This implies that $g(T/2) \geqslant g(T)$. Therefore,

$$\big(v(0), v(T)\big) = \big\|v(T/2)\big\|^2 \geqslant \big\|v(T)\big\|^2.$$

It follows now from (3.1) and the positivity of α that

$$\|f\|^2 \geqslant \alpha^2\big\|v(0)\big\|^2 + (2\alpha+1)\big\|v(T)\big\|^2.$$

The lemma is proved. □

The following result is straightforward.

Lemma 2. *If x, y are nonnegative numbers and q is a positive number, then*

$$x + qy \geqslant (q+1)x^{1/(q+1)}y^{q/(q+1)}.$$

Lemma 3. *If $v(t)$ is a solution of* (1.3), *then*

$$\big\|v(t)\big\| \leqslant H(t,\alpha)\alpha^{t/T-1}\|f\|, \quad \forall t \in (0, T).$$

Proof. Applying Lemma 2 and Lemma 1 with

$$\begin{aligned} x &= (2\alpha+1)\big\|v(T)\big\|^2, \\ q &= \frac{1-t/T}{t/T}, \\ y &= \frac{t/T}{1-t/T}\alpha^2\big\|v(0)\big\|^2, \end{aligned}$$

we get

$$\begin{aligned} \|f\|^2 &\geqslant (2\alpha+1)\big\|v(T)\big\|^2 + \alpha^2\big\|v(0)\big\|^2 \\ &= x + qy \\ &\geqslant (q+1)x^{\frac{1}{q+1}}y^{\frac{q}{q+1}} \\ &= \frac{T}{t}\big((2\alpha+1)\big\|v(T)\big\|^2\big)^{t/T}\left(\frac{t}{T-t}\alpha^2\big\|v(0)\big\|^2\right)^{1-t/T} \\ &= \left(\frac{1}{H(t,\alpha)}\alpha^{(1-t/T)}\big\|v(T)\big\|^{t/T}\big\|v(0)\big\|^{1-t/T}\right)^2. \end{aligned} \tag{3.2}$$

By the method of log-convexity we have

$$\big\|v(T)\big\|^{t/T}\big\|v(0)\big\|^{1-t/T} \geqslant \big\|v(t)\big\|, \quad \forall t \in [0, T]. \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$\|f\|^2 \geqslant \left(\frac{1}{H(t,\alpha)}\alpha^{1-t/T}\big\|v(t)\big\|\right)^2.$$

Hence

$$\big\|v(t)\big\| \leqslant H(t,\alpha)\alpha^{t/T-1}\|f\|, \quad \forall t \in [0, T].$$

The lemma is proved. □

Denote by w the solution of the problem

$$\begin{cases} w_t + Aw = 0, & 0 < t < T, \\ \alpha w(0) + w(T) = u(T). \end{cases} \tag{3.4}$$

Lemma 4. *The following inequality holds*

$$\big\|v(t) - w(t)\big\| \leqslant H(t,\alpha)\alpha^{t/T-1}\epsilon, \quad \forall t \in [0, T].$$

Proof. Set $\omega(t) = v(t) - w(t)$, $\forall t \in [0, T]$. Then $\omega(t)$ is the solution of the problem

$$\begin{cases} \omega_t + A\omega = 0, & 0 < t < T, \\ \alpha\omega(0) + \omega(T) = f - u(T). \end{cases}$$

In virtue of Lemma 3,

$$\|\omega(t)\| \leqslant H(t,\alpha)\alpha^{t/T-1}\|f - u(T)\| \leqslant H(t,\alpha)\alpha^{t/T-1}\epsilon, \quad \forall t \in [0, T].$$

Thus, the lemma is proved. □

Lemma 5. *The following inequality holds*

$$\|w(t) - u(t)\| \leqslant H(t,\alpha)\alpha^{t/T}E, \quad \forall t \in [0, T].$$

Proof. Set $z(t) := u(t) - w(t)$, $\forall t \in [0, T]$. Then, by direct calculation,

$$\alpha z(0) + z(T) = \alpha u(0).$$

Thus, $z(t)$ is the solution of the problem

$$\begin{cases} z_t + Az = 0, & 0 < t < T, \\ \alpha z(0) + z(T) = \alpha u(0). \end{cases} \tag{3.5}$$

Using Lemma 3, we get

$$\|z(t)\| \leqslant H(t,\alpha)\alpha^{t/T-1}\|\alpha u(0)\| \leqslant H(t,\alpha)\alpha^{t/T}E, \quad \forall t \in [0, T].$$

The lemma is proved. □

Remark 6. From Lemmas 4 and 5 and the triangle inequality we arrive at the propositions of the theorem with $Q(t,\alpha)$ replaced by $H(t,\alpha)$.

It is well known that, for all $t \in [0, T]$,

$$u(t) = \sum_{n=1}^{+\infty} e^{(T-t)\lambda_n}\big(u(T), \phi_n\big)\phi_n, \tag{3.6}$$

$$v(t) = \sum_{n=1}^{+\infty} \frac{e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}}(f, \phi_n)\phi_n, \tag{3.7}$$

$$w(t) = \sum_{n=1}^{+\infty} \frac{e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}}\big(u(T), \phi_n\big)\phi_n. \tag{3.8}$$

Lemma 6. *If $v(t)$ is a solution of* (1.3)*, then*

$$\|v(t)\| \leqslant K(t)\alpha^{t/T-1}\|f\|, \quad \forall t \in [0, T].$$

Proof. The representation (3.7) implies that $\|v(0)\| \leqslant \frac{1}{\alpha}\|f\|$ and $\|v(T)\| \leqslant \|f\|$. Now, for all $t \in (0, T)$, applying Lemma 2 with

$$x = \alpha^{t/T},$$
$$q = \frac{t/T}{1 - t/T} = \frac{t}{T - t},$$
$$y = \frac{1 - t/T}{t/T}\alpha^{(t/T-1)}e^{-\lambda_n T},$$

we have

$$\begin{aligned} \alpha^{t/T} + \alpha^{(t/T-1)}e^{-\lambda_n T} &= x + qy \geqslant (q+1)x^{\frac{1}{q+1}}y^{\frac{q}{q+1}} \\ &= \left(\frac{1}{1 - t/T}\right)\alpha^{t/T(1-t/T)}\left(\frac{1 - t/T}{t/T}\right)^{t/T}\alpha^{(t/T-1)t/T}\big(e^{-\lambda_n T}\big)^{t/T} \\ &= \left(\frac{1}{1 - t/T}\right)\left(\frac{1 - t/T}{t/T}\right)^{t/T}e^{-\lambda_n t} \end{aligned}$$

$$= \frac{1}{(1-t/T)^{(1-t/T)}(t/T)^{t/T}} e^{-\lambda_n t}$$
$$= \frac{1}{K(t)} e^{-\lambda_n t}.$$

Therefore, we obtain

$$\frac{e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}} \leqslant K(t)\alpha^{t/T-1}$$

for all $t \in [0, T]$ and $n = 1, 2, \ldots$. This implies that

$$\|v(t)\| \leqslant K(t)\alpha^{t/T-1}\|f\|, \quad \forall t \in [0, T].$$

The lemma is proved. □

Lemma 7. *The following inequality holds*

$$\|v(t) - w(t)\| \leqslant K(t)\alpha^{t/T-1}\epsilon, \quad \forall t \in [0, T].$$

Proof. Similar to that of Lemma 4. □

Lemma 8. *The following inequality holds*

$$\|u(t) - w(t)\| \leqslant K(t)\alpha^{t/T}E, \quad \forall t \in [0, T].$$

Proof. We have

$$\begin{aligned}
\|u(t) - w(t)\|^2 &= \sum_{n=1}^{+\infty}\left(e^{(T-t)\lambda_n} - \frac{e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}}\right)^2 \big(u(T), \phi_n\big)^2 \\
&= \alpha^2 \sum_{n=1}^{+\infty}\left(\frac{e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}}\right)^2 e^{2\lambda_n T}\big(u(T), \phi_n\big)^2 \\
&\leqslant \alpha^2 \sum_{n=1}^{+\infty}\big(K(t)\alpha^{t/T-1}\big)^2 e^{2\lambda_n T}\big(u(T), \phi_n\big)^2 \\
&= \big(K(t)\alpha^{t/T}\big)^2 \sum_{n=1}^{+\infty} e^{2\lambda_n T}\big(u(T), \phi_n\big)^2 \\
&= \big(K(t)\alpha^{t/T}\big)^2 \|u(0)\|^2 \leqslant \big(K(t)\alpha^{t/T}E\big)^2.
\end{aligned}$$

The lemma is proved. □

The proposition of the theorem follows now immediately from Lemmas 4, 5, 7 and 8 and the triangle inequality.

Proof of Remark 2. With

$$\alpha = \alpha_2 := \frac{\epsilon}{E}\frac{1-t/T}{t/T},$$

we obtain for all $t \in (0, T)$ that

$$\begin{aligned}
\|v(t) - u(t)\| &\leqslant Q(t, \alpha_2)\big(\alpha_2^{t/T-1}\epsilon + \alpha_2^{t/T}E\big) \\
&\leqslant K(t)\big(\alpha_2^{t/T-1}\epsilon + \alpha_2^{t/T}E\big) \\
&= \epsilon^{t/T}E^{1-t/T}K(t)\left\{\left(\frac{1-t/T}{t/T}\right)^{t/T-1} + \left(\frac{1-t/T}{t/T}\right)^{t/T}\right\} \\
&= \epsilon^{t/T}E^{1-t/T}K(t)\left(\frac{1-t/T}{t/T}\right)^{t/T}\left\{\left(\frac{1-t/T}{t/T}\right)^{-1} + 1\right\} \\
&= \epsilon^{t/T}E^{1-t/T}K(t)\left(\frac{1-t/T}{t/T}\right)^{t/T}\frac{1}{1-t/T} \\
&= \epsilon^{t/T}E^{1-t/T}K(t)\left(\frac{1}{t/T}\right)^{t/T}\frac{(1-t/T)^{t/T}}{1-t/T}
\end{aligned}$$

$$
\begin{aligned}
&= \epsilon^{t/T} E^{1-t/T} K(t) \frac{1}{(t/T)^{t/T}(1-t/T)^{1-t/T}} \\
&= \epsilon^{t/T} E^{1-t/T} K(t) \frac{1}{K(t)} \\
&= \epsilon^{t/T} E^{1-t/T}.
\end{aligned}
$$

The remark is proved. □

3.2. Proof of Theorem 2

From (3.6) and (3.8), we have

$$
\begin{aligned}
\|u(t)-w(t)\|^2 &= \left\| \sum_{n=1}^{+\infty} \left(e^{(T-t)\lambda_n} - \frac{e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}} \right) \big(u(T),\phi_n\big)\phi_n \right\|^2 \\
&= \left\| \sum_{n=1}^{+\infty} \frac{\alpha e^{\lambda_n (T-t)}}{\alpha + e^{-\lambda_n T}} \big(u(T),\phi_n\big)\phi_n \right\|^2 \\
&= \left\| \sum_{n=1}^{+\infty} \frac{\alpha e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}} \big(u(0),\phi_n\big)\phi_n \right\|^2 \\
&= \sum_{n=1}^{+\infty} \left(\frac{\alpha e^{-\lambda_n t}}{\alpha + e^{-\lambda_n T}} \right)^2 \big(u(0),\phi_n\big)^2 \\
&= \alpha^{2\frac{t}{T}} \sum_{n=1}^{+\infty} \frac{e^{-2\lambda_n t}}{(\alpha + e^{-\lambda_n T})^{2\frac{t}{T}}} \left(\frac{\alpha}{\alpha + e^{-\lambda_n T}} \right)^{2(1-\frac{t}{T})} \big(u(0),\phi_n\big)^2 \\
&\leqslant \alpha^{2\frac{t}{T}} \sum_{n=1}^{+\infty} \left(\frac{\alpha}{\alpha + e^{-\lambda_n T}} \right)^{2(1-\frac{t}{T})} \big(u(0),\phi_n\big)^2 \\
&= \alpha^{2\frac{t}{T}} \sum_{n=1}^{+\infty} \left(\frac{\alpha}{\alpha \lambda_n^{\frac{\beta T}{T-t}} + \lambda_n^{\frac{\beta T}{T-t}} e^{-\lambda_n T}} \right)^{2(1-\frac{t}{T})} \lambda_n^{2\beta} \big(u(0),\phi_n\big)^2 \\
&\leqslant \alpha^{2\frac{t}{T}} \sum_{n=1}^{+\infty} \left(\frac{\alpha}{\alpha \lambda_n^{\frac{\beta T}{T-t}} + \lambda_1^{\frac{\beta T}{T-t}} e^{-\lambda_n T}} \right)^{2(1-\frac{t}{T})} \lambda_n^{2\beta} \big(u(0),\phi_n\big)^2.
\end{aligned} \tag{3.9}
$$

Case $\boldsymbol{\beta(t) = \frac{\beta T}{T-t} \geqslant 1}$.

Since $\beta(t) \geqslant 1$, we have

$$
\begin{aligned}
\alpha \lambda_n^{\beta(t)} + \lambda_1^{\beta(t)} e^{-\lambda_n T} &= \big(\alpha^{1/\beta(t)}\lambda_n\big)^{\beta(t)} + \big(\lambda_1 e^{(-\lambda_n T)/\beta(t)}\big)^{\beta(t)} \\
&\geqslant 2 \left(\frac{\alpha^{1/\beta(t)}\lambda_n + \lambda_1 e^{(-\lambda_n T)/\beta(t)}}{2} \right)^{\beta(t)} \\
&= 2^{1-\beta(t)} \big(\alpha^{1/\beta(t)}\lambda_n + \lambda_1 e^{(-\lambda_n T)/\beta(t)}\big)^{\beta(t)}.
\end{aligned}
$$

Case $\boldsymbol{0 < \beta(t) = \frac{\beta T}{T-t} < 1}$.

Since $\beta(t) \in (0,1)$, we have

$$
\alpha \lambda_n^{\beta(t)} + \lambda_1^{\beta(t)} e^{-\lambda_n T} = \big(\alpha^{\frac{1}{\beta(t)}}\lambda_n\big)^{\beta(t)} + \big(\lambda_1 e^{(-\lambda_n T)/\beta(t)}\big)^{\beta(t)} \geqslant \big(\alpha^{1/\beta(t)}\lambda_n + \lambda_1 e^{(-\lambda_n T)/\beta(t)}\big)^{\beta(t)}.
$$

As

$$
C(t) = \begin{cases} 1, & \text{if } 0 < \beta(t) < 1, \\ 2^{1-\beta(t)}, & \text{if } \beta(t) \geqslant 1, \end{cases}
$$

we obtain

$$
\alpha \lambda_n^{\beta(t)} + \lambda_1^{\beta(t)} e^{-\lambda_n T} \geqslant C(t) \big(\alpha^{1/\beta(t)}\lambda_n + \lambda_1 e^{(-\lambda_n T)/\beta(t)}\big)^{\beta(t)}. \tag{3.10}
$$

Let $g(\lambda) = (\alpha^{1/\beta(t)}\lambda + \lambda_1 e^{(-\lambda T)/\beta(t)})^{-1}$. Then

$$\sup_{\lambda \geqslant \lambda_1} g(\lambda) \leqslant g\left(\ln\left(\left(\frac{T\lambda_1}{\beta(t)}\right)^{\beta(t)} \Big/ \alpha\right) \Big/ T\right) = \frac{T}{\alpha^{1/\beta(t)} \ln((\frac{T\lambda_1 e}{\beta(t)})^{\beta(t)}/\alpha)}, \quad \text{if } 0 < \alpha < \left(\frac{T\lambda_1}{\beta(t)}\right)^{\beta(t)} \tag{3.11}$$

and

$$\sup_{\lambda \geqslant \lambda_1} g(\lambda) = g(\lambda_1) < \left(\lambda_1 e^{(-\lambda_1 T)/\beta(t)}\right)^{-1} = \frac{e^{(\lambda_1 T)/\beta(t)}}{\lambda_1}, \quad \text{if } \alpha \geqslant \left(\frac{T\lambda_1}{\beta(t)}\right)^{\beta(t)}. \tag{3.12}$$

From (3.9)–(3.11) and (3.12) we have

$$\|u(t) - w(t)\| \leqslant \begin{cases} \alpha^{\frac{t}{T}} (\frac{T}{\ln((\frac{T\lambda_1 e}{\beta(t)})^{\beta(t)}/\alpha)})^{\beta(t)(1-\frac{t}{T})} C(t)^{\frac{t}{T}-1} E_1, & \text{if } 0 < \alpha < (\frac{T\lambda_1}{\beta(t)})^{\beta(t)}, \\ \alpha (\frac{e^{\lambda_1 T}}{\lambda_1^{\beta(t)} C(t)})^{1-\frac{t}{T}} E_1, & \text{if } \alpha \geqslant (\frac{T\lambda_1}{\beta(t)})^{\beta(t)}. \end{cases} \tag{3.13}$$

The proposition of the theorem follows now immediately from Lemma 4, (3.13) and the triangle inequality.

Proof of Remark 5.

Case $0 < \beta \leqslant 1$.

Using Theorem 2, we have

$$\|u(0) - v(0)\| \leqslant \frac{\epsilon}{\alpha} + \left(\frac{T}{\ln(c/\alpha)}\right)^{\beta} E_1, \tag{3.14}$$

where $c = (T\lambda_1 e/\beta)^\beta$. If choose $\alpha = \frac{\epsilon}{E_1}(\frac{1}{T}\ln\frac{E_1}{\epsilon})^{\beta+\delta}$, then

$$\frac{\epsilon}{\alpha} = E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\delta} = E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta} o(1) \quad \text{as } \epsilon \to 0, \tag{3.15}$$

since

$$\lim_{\epsilon \to 0^+}\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\delta} = 0.$$

On the other hand, as $\epsilon \to 0$,

$$\begin{aligned}\left(\frac{T}{\ln(c/\alpha)}\right)^{\beta} E_1 &= \left(\frac{T}{\ln(c\frac{E_1}{\epsilon}(\frac{1}{T}\ln\frac{E_1}{\epsilon})^{-(\beta+\delta)})}\right)^{\beta} E_1 \\ &= E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\left(\frac{\ln\frac{E_1}{\epsilon}}{\ln\frac{E_1}{\epsilon} + \ln c - (\beta+\delta)\ln(\frac{1}{T}\ln\frac{E_1}{\epsilon})}\right)^{\beta} \\ &= E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\left(1 + o(1)\right)\end{aligned} \tag{3.16}$$

since, by direct inspection,

$$\lim_{\epsilon \to 0^+} \frac{\ln\frac{E_1}{\epsilon}}{\ln\frac{E_1}{\epsilon} + \ln c - (\beta+\delta)\ln(\frac{1}{T}\ln\frac{E_1}{\epsilon})} = 1.$$

Now, from (3.14), (3.15) and (3.16) we conclude that

$$\|u(0) - v(0)\| \leqslant E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\left(1 + o(1)\right).$$

Case $\beta > 1$.

From (3.9) we have

$$\|w(0) - u(0)\|^2 \leqslant \left(\sup_{x>0} g_\alpha(x)\right)^2 \sum_{n=1}^{+\infty} \lambda_n^{2\beta}\left(u(0), \phi_n\right)^2 \leqslant \left(\sup_{x>0} g_\alpha(x)\right)^2 E_1^2, \tag{3.17}$$

where

$$g_\alpha(x) := \frac{\alpha}{\alpha x^\beta + \lambda_1^\beta e^{-xT}}, \quad x > 0.$$

Taking the derivative of $g_\alpha(x)$ with respect to x and letting it equal zero, we get

$$e^{-xT} = \frac{\alpha\beta}{T\lambda_1^\beta} x^{\beta-1}. \tag{3.18}$$

Since $\beta > 1$, there exists a unique solution x_α of (3.18). Further, by checking the sign of $g'_\alpha(x)$, we obtain

$$\sup_{x>0} g_\alpha(x) = g_\alpha(x_\alpha) = \frac{\alpha}{\alpha x_\alpha^\beta + \lambda_1^\beta e^{-x_\alpha T}} < \left(\frac{1}{x_\alpha}\right)^\beta. \tag{3.19}$$

As $\beta > 1$ and (3.18), we find that if α tends to zero, then x_α tends to infinity and from that it can be deduced that

$$\lim_{\alpha\to 0^+} \frac{\ln(T/\alpha)}{x_\alpha T} = 1. \tag{3.20}$$

From Lemma 7, the triangle inequality, (3.17) and (3.19) we claim that

$$\left\|u(0) - v(0)\right\| \leqslant \frac{\epsilon}{\alpha} + \left(\frac{1}{x_\alpha}\right)^\beta E_1 = \frac{\epsilon}{\alpha} + \left(\frac{T}{\ln(T/\alpha)}\right)^\beta E_1 \left(\frac{\ln(T/\alpha)}{x_\alpha T}\right)^\beta. \tag{3.21}$$

With $\alpha = \frac{\epsilon}{E_1}(\frac{1}{T}\ln\frac{E_1}{\epsilon})^{\beta+\delta}$, as $\epsilon \to 0$, we get

$$\frac{\epsilon}{\alpha} = E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\delta} = E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta} o(1) \tag{3.22}$$

since

$$\lim_{\epsilon\to 0^+}\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\delta} = 0.$$

On the other hand, with $\alpha = \frac{\epsilon}{E_1}(\frac{1}{T}\ln\frac{E_1}{\epsilon})^{\beta+\delta}$, as $\epsilon \to 0$ we have

$$\begin{aligned}\left(\frac{T}{\ln T/\alpha}\right)^\beta E_1\left(\frac{\ln(T/\alpha)}{x_\alpha T}\right)^\beta &= E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\left(\frac{\ln\frac{E_1}{\epsilon}}{\ln\frac{E_1}{\epsilon} + \ln T - (\beta+\delta)\ln(\frac{1}{T}\ln\frac{E_1}{\epsilon})}\right)^\beta\left(\frac{\ln(T/\alpha)}{x_\alpha T}\right)^\beta \\ &= E_1\left[\frac{1}{T}\ln\frac{E_1}{\epsilon}\right]^{-\beta}\big(1 + o(1)\big)\end{aligned} \tag{3.23}$$

since

$$\lim_{\epsilon\to 0^+}\frac{\ln\frac{E_1}{\epsilon}}{\ln\frac{E_1}{\epsilon} + \ln T - (\beta+\delta)\ln(\frac{1}{T}\ln\frac{E_1}{\epsilon})} = 1$$

and by (3.20)

$$\lim_{\epsilon\to 0^+}\frac{\ln(T/\alpha)}{x_\alpha T} = \lim_{\alpha\to 0^+}\frac{\ln(T/\alpha)}{x_\alpha T} = 1.$$

From (3.21)–(3.23) it follows (2.5).

Now, we prove (2.6). By the same method of proving (2.5) we obtain that

$$\left\|u(t) - w(t)\right\| \leqslant K(t)\alpha^{t/T}\left(\frac{T}{\ln\frac{1}{\alpha}}\right)^\beta E_1\big(1 + o(1)\big) \quad \text{as } \epsilon \to 0. \tag{3.24}$$

From Lemma 7, the triangle inequality, and (3.24) we obtain

$$\left\|u(t) - v(t)\right\| \leqslant K(t)\alpha^{t/T-1}\epsilon + K(t)\alpha^{t/T}\left(\frac{T}{\ln\frac{1}{\alpha}}\right)^\beta\big(1 + o(1)\big), \quad \forall t \in (0, T).$$

Choosing $\alpha = \frac{1-t/T}{t/T}\frac{\epsilon}{E}[\frac{1}{T}\ln\frac{E_1}{\epsilon}]^\beta$, we arrive at (2.6). $\square$

3.3. Proof of Theorem 3

We have

$$\alpha^{2t/T}\sum_{n=1}^{+\infty}\left(\frac{\alpha}{\alpha+e^{-\lambda_n T}}\right)^{2(1-t/T)}\big(u(0),\phi_n\big)^2=\alpha^{2t/T}\sum_{n=1}^{+\infty}\left(\frac{\alpha}{\alpha p_n(t)+p_n(t)e^{-\lambda_n T}}\right)^{2(1-t/T)}e^{2\beta\lambda_n}\big(u(0),\phi_n\big)^2,$$

where $p_n(t)=e^{\beta\lambda_n\frac{T}{T-t}}$, $\forall t\in[0,T)$. If $\beta\geqslant T-t$, then $p_n(t)e^{-\lambda_n T}=e^{\lambda_n T(\frac{\beta}{T-t}-1)}\geqslant e^0=1$. Therefore, $\frac{\alpha}{\alpha p_n(t)+p_n(t)e^{-\lambda_n T}}<\alpha$. Hence

$$\big\|u(t)-w(t)\big\|\leqslant\alpha E_2. \tag{3.25}$$

If $0<\beta<T-t$, then applying Lemma 2 with $x=\alpha p_n(t)$, $q=\beta/(T-t-\beta)$, $y=p_n(t)e^{-\lambda_n T}(T-t-\beta)/\beta$, we obtain

$$\begin{aligned}\alpha p_n(t)+p_n(t)e^{-\lambda_n T}&=x+qy\geqslant(q+1)x^{1/(q+1)}y^{q/(q+1)}\\&=(T-t)/(T-t-\beta)\left(\frac{T-t-\beta}{\beta}\right)^{\beta/(T-t)}\alpha^{(T-t-\beta)/(T-t)}.\end{aligned}$$

Therefore,

$$\left(\frac{\alpha}{\alpha p_n(t)+p_n(t)e^{-\lambda_n T}}\right)^{(1-t/T)}\leqslant\alpha^{\beta/T}\left(1-\frac{\beta}{T-t}\right)^{(T-t-\beta)/T}\left(\frac{\beta}{T-t}\right)^{\beta/T}<\alpha^{\beta/T}.$$

This implies that

$$\big\|u(t)-w(t)\big\|\leqslant\alpha^{(t+\beta)/T}. \tag{3.26}$$

The proposition of the theorem follows now immediately from Lemma 4, Lemma 7, (3.25), (3.26), and the triangle inequality.

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